RECONSTRUCTION OF A PLANE CONVEX BODY FROM THE CURVATURE OF ITS BOUNDARY†

BY

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ABSTRACT

Let $\tilde{K}(w)$ denote the class of plane convex bodies having a width function w, where w' is absolutely continuous. It is proved that a body in $\tilde{K}(w)$ is determined (up to translation) by the radius of curvature function of its boundary. This result is then used for a characterization of the extreme (indecomposable) bodies in $\tilde{K}(w)$ and for a density theorem for Reuleaux polygons in $\tilde{K}(1)$.

1.

For a plane convex body K, $u \in E^2$ and real θ, let $H(K,u) = \sup\{\langle u,x\rangle: x \in K\}$ be the support function of K, and let $f_K(\theta) = H(K,u_\theta)$ where $u_\theta = (\cos\theta, \sin\theta)$. The support function is continuous, convex, and positively homogeneous. The homogeneity implies that a support function is completely determined by its restriction to the unit circle, hence there is a one-to-one correspondence between the class of plane convex bodies and a class $\tilde{F}$ of continuous, 2π periodic real functions. This correspondence preserves Minkowski addition and multiplication by nonnegative numbers, that is, $f_{K+L}(\theta) = f_K(\theta) + f_L(\theta)$ and $f_{\lambda K}(\theta) = \lambda f_K(\theta)$ for convex bodies K, L, and $\lambda \geqq 0$. The following is a characterization of the class $\tilde{F}$.

DEFINITION. A real function $f(\theta)$ is said to be circle convex if for all real h such that $|h| \leqq \frac{1}{2}\pi$ we have:

$$(1) \qquad f(\theta + h) + f(\theta - h) \geqq 2f(\theta)\cos h.$$

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THEOREM 1. *A real function f is in $\tilde{F}$ iff it is continuous, 2π periodic, and circle convex.*

PROOF. If $f \in \tilde{F}$ then $f(\theta) = H(K, u_\theta)$ for some convex body K, and f is clearly continuous and 2π periodic. By the convexity and homogeneity of H it follows for $|\theta - \varphi| < \pi$, that

$$\begin{aligned} f(\theta) + f(\varphi) &= H(u_\theta) + H(u_\varphi) \geqq H(u_\theta + u_\varphi) \\ &= \| u_\theta + u_\varphi \| H((u_\theta + u_\varphi)/\| u_\theta + u_\varphi \|) = \| u_\theta + u_\varphi \| f(\tfrac{1}{2}(\theta + \varphi)). \end{aligned}$$

But

$$\begin{aligned} \| u_\theta + u_\varphi \| &= \langle u_\theta + u_\varphi, u_\theta + u_\varphi \rangle^{\frac{1}{2}} = (2 + 2\langle u_\theta, u_\varphi \rangle)^{\frac{1}{2}} \\ &= (2(1 + \cos(\theta - \varphi))^{\frac{1}{2}} = 2\cos(\tfrac{1}{2}(\theta - \varphi)). \end{aligned} \tag{2}$$

Thus $f(\theta) + f(\varphi) \geqq 2 f(\frac{1}{2}(\theta + \varphi)) \cos(\frac{1}{2}(\theta - \varphi))$ which is another form of (1) proved for $|h| < \frac{1}{2}\pi$. By continuity (1) is also true for $|h| = \frac{1}{2}\pi$.

Conversely, suppose f is continuous and 2π periodic, and define $H(u) = \| u \| f(\theta_u)$ for $u \neq 0$, where $u = \| u \| (\cos\theta_u, \sin\theta_u)$, and $H(0) = 0$. The function H is clearly continuous and positively homogeneous. If H is also convex then H is the support function of some convex body (see [4, p. 57]). Thus we only have to show that if H is not convex then f is not circle convex.

If H is not convex there are $u, v \in E^2$ and $0 < \lambda < 1$ such that

$$H(\lambda u + (1 - \lambda)v) > \lambda H(u) + (1 - \lambda)H(v). \tag{3}$$

H is continuous, hence there is some neighborhood of λ for which (3) still holds. There are numbers $0 \leqq \alpha < \beta \leqq 1$ such that (3) holds for every $\alpha < \lambda < \beta$, with equality for $\lambda = \alpha$ and $\lambda = \beta$. Let $u_1 = \beta u + (1 - \beta)v$ and $v_1 = \alpha u + (1 - \alpha)v$. It is easily checked that for arbitrary $0 < \mu < 1$,

$$H(\mu u_1 + (1 - \mu)v_1) > \mu H(u_1) + (1 - \mu) H(v_1).$$

Hence (replacing u by u_1 and v by v_1) we may assume that (3) holds for every $0 < \lambda < 1$. $H(0) = 0$, hence $u \neq 0$ and $v \neq 0$ (otherwise we would have equality in (3)). With $\lambda = \| v \| / (\| u \| + \| v \|)$ in (3) we have

$$H\left(\frac{\| v \| u}{\| u \| + \| v \|} + \frac{\| u \| v}{\| u \| + \| v \|}\right) > \frac{\| v \|}{\| u \| + \| v \|} H(u) + \frac{\| u \|}{\| u \| + \| v \|} H(v).$$

$\| v \| u / (\| u \| + \| v \|)$ and $\| u \| v / (\| u \| + \| v \|)$ have the same length, hence their sum bisects the angle between them, and its length is

$$\left\| \frac{\|v\|u}{\|u\|+\|v\|} + \frac{\|u\|v}{\|u\|+\|v\|} \right\| = 2\,\frac{\|u\|\;\|v\|}{\|u\|+\|v\|}\cos(\tfrac{1}{2}(\theta_u-\theta_v))$$

(see (2)). Now (3) takes the form

$$2\,\frac{\|u\|\;\|v\|}{\|u\|+\|v\|}\cos(\tfrac{1}{2}(\theta_u-\theta_v))f(\tfrac{1}{2}(\theta_u+\theta_v)) > \frac{\|u\|\;\|v\|}{\|u\|+\|v\|}(f(\theta_u)+f(\theta_v)),$$

hence
$$2f(\tfrac{1}{2}(\theta_u+\theta_v))\cos(\tfrac{1}{2}(\theta_u-\theta_v)) > f(\theta_u)+f(\theta_v)$$

and f is not circle convex. Q.E.D.

We shall from now on refer to the function f_K in $\tilde{F}$ as the support function of K.

2.

DEFINITION. Let K be a plane convex body. The width function w_K of K is $w_K(\theta) = f_K(\theta) + f_K(\theta+\pi)$ (which is the support function of $K + (-K)$).

For a given function w we denote by $\tilde{K}(w)$ the class of plane convex bodies whose width function equals w.

DEFINITION. Let AC^1 be the class of all 2π periodic functions $f: R \to R$ such that f' is absolutely continuous on every bounded interval. If $f \in AC^1$ we shall sometimes say that f is AC^1. A function is AC^1 on a set $D \subseteq R$ if f' is absolutely continuous in the set D.

THEOREM 2. *If $K \in \tilde{K}(w)$ and $w \in AC^1$, then $f_K \in AC^1$.*

PROOF. From [4, pp. 56–7] it is clear that f_K' exists and is continuous iff the boundary of K contains no line segments. If there were a line segment in the boundary of K, then there would also be one in the boundary of $K + (-K)$, and w', the derivative of the support function of $K + (-K)$, would fail to exist somewhere. But $w \in AC^1$, hence f_K' exists everywhere.

Let $F_1(t) = H(K,(t,1))$ and $F_2(t) = H(K,(t,-1))$, where $H(K,x)$ is the support function of K. Since $f_K(\theta) = (-1)^{i-1}\sin\theta\, F_i(\cot\theta)$ for $(i-1)\pi < \theta < i\pi$, for $i = 1, 2$, the derivatives F_i' exist within these intervals and are both non-decreasing, by the convexity of F_i.

The fact that $K \in \tilde{K}(w)$ is expressed by $F_1(t) + F_2(-t) = g(t)$, where $g(t) = (1+t^2)^{\frac{1}{2}} w(\operatorname{arc\,cot} t)$ (see [9]).

For an arbitrary finite set $\{(a_i,b_i)\}_{i=1}^n$ of disjoint subintervals of $[-1,1]$ we have

$$\sum_{i=1}^{n} |F_1'(b_i) - F_1'(a_i)| + \sum_{i=1}^{n} |F_2'(b_i) - F_2'(a_i)|$$

(4)
$$= \sum_{i=1}^{n} F_1'(b_i) + F_2'(b_i) - F_1'(a_i) - F_2'(a_i) = \sum_{i=1}^{n} g'(b_i) - g'(a_i)$$

$$= \sum_{i=1}^{n} |g'(b_i) - g'(a_i)|.$$

It is therefore clear that if g' is AC^1 on $[-1,1]$ then so are F_1 and F_2. But g is AC^1 on $[-1, 1]$ since

(5)
$$g'(t) = t(1+t^2)^{-\frac{1}{2}} w(\operatorname{arc}\cot t) - (1+t^2)^{-\frac{1}{2}} w'(\operatorname{arc}\cot t),$$

and sums and products of AC^1 functions and a composition of an AC^1 function and a monotone AC^1 function are AC^1 (see [6, I, p. 245]), hence f is AC^1 on $\pi/4 \leqq \theta \leqq 3\pi/4$ and $5\pi/4\,\theta \leqq 7\pi/4$. In the same manner (using the lines $(t, 1)$ and $(t, -1)$) we prove that $f \in AC^1$ all over the interval $[0, 2\pi]$.

3.

The radius of curvature of a curve C is usually defined in terms of the second derivatives of a parametric representation of C. Here we use a geometric definition of the radius of curvature which is applicable to any curve which is the boundary of a convex body, without any smoothness assumptions.

DEFINITION. Let f_K be the support function of a plane convex body K, and let θ be a real number. If there exists a circle C of radius R such that

(6)
$$f_K(\theta + h) - f_C(\theta + h) = o(h^2)$$

then R is said to be the radius of curvature of K in the direction θ, and is denoted by $R_K(\theta)$. (C may be a point, and in that case $R = 0$.)

DEFINITION. The function $R_f(\theta) = f''(\theta) + f(\theta)$ is called the radius of curvature of f. We have seen that if $K \in \tilde{K}(w)$, where $w \in AC^1$, then R_{f_K} is defined a.e.. A slight change in (1) yields:

(7)
$$f_K(\theta + h) + f_K(\theta - h) - 2f_K(\theta) \geqq 2f_K(\theta)(\cos h - 1) = -4f_K(\theta)\sin^2 \tfrac{1}{2}h,$$

or:

(8)
$$(f_K(\theta + h) + f_K(\theta - h) - 2f_K(\theta))/h^2 \geqq -f_K(\theta)\sin^2 \tfrac{1}{2}h/(\tfrac{1}{2}h)^2.$$

It is easily seen that if $f''(\theta)$ exists then the left-hand term of (8) tends to $f''(\theta)$ as h tends to zero (see [6, II, p. 37]). Hence,

$$f''_K(\theta) \geqq -f_K(\theta) \quad \text{or}$$

$$R_{f_K}(\theta) \geqq 0. \tag{9}$$

THEOREM 3. *Let K be a convex body. If f''_K exists at θ then K has a radius of curvature R in the direction θ, and $R = R_{f_K}(\theta)$.*

PROOF. If $f_K(\theta)$ exists then we have

$$f_K(\theta + h) = f_K(\theta) + hf'_K(\theta) + \tfrac{1}{2}h^2 f''_K(\theta) + o(h^2). \tag{10}$$

The support function of a circle C of radius $R \geqq 0$ and center a is

$$f_C(\varphi) = R + \langle a, u_\varphi \rangle$$

and $f''_C(\theta) = -\langle a, u_\theta \rangle$. If f_C is to be a second-order approximation of f_K at θ we must have, by (10), $f_K(\theta) = R + \langle a, u_\theta \rangle$ and $f''_K(\theta) = -\langle a, u_\theta \rangle$. Thus $R = f_K(\theta) + f''_K(\theta)$ and, by (9), R is nonnegative and is the radius of a circle. By equating the first and second derivatives of f_K and f_C at θ we obtain a system of two linear equations with a unique solution for the components of a. Hence R is the radius of curvature of K in the direction θ.

THEOREM 4. *Let w be an AC^1 width function of a plane convex body.*

I. *The radius of curvature function $R(\theta)$ of a convex body $K \in \tilde{K}(w)$ satisfies the following conditions:*

(i) *R is nonnegative and measurable.*

(ii) $$R(\theta) + R(\theta + \pi) = w''(\theta) + w(\theta) \text{ a.e.}.$$

(iii) $$\int_0^\pi R(\theta) \sin d\theta = w(0).$$

(iv) $$\int_0^\pi R(\theta) \cos\theta \, d\theta = -w'(0).$$

II. *Let $R(\theta)$ be a real function defined on the real line, satisfying conditions* (i)-(iv) *of Part* I. *Then*

(i) *there exists a convex body $K \in \tilde{K}(w)$ such that $R(\theta) = R_K(\theta)$* a.e.

(ii) *K is unique, up to translation, that is, if $g(\theta)$ is an AC^1 function and $R_g(\theta) = R(\theta)$ a.e. then g is the support function of a translate of K.*

PROOF. We shall first prove Part II.

Define $f(\theta) = \int_0^\theta R(\sigma) \sin(\theta - \sigma) d\sigma$ (see [2, p. 115]).

(a) *f circle convex.* Both sides of (1) are even functions of h, so it suffices to prove (1) for $0 \leqq h \leqq \frac{1}{2}\pi$.

$$f(\theta+h)+f(\theta-h) = \int_0^{\theta+h} R(\sigma)\sin(\theta+h-\sigma)d\sigma + \int_0^{\theta-h} R(\sigma)\sin(\theta-h-\sigma)d\sigma$$

$$= \int_0^{\theta} R(\sigma)(\sin(\theta+h-\sigma)+\sin(\theta-h-\sigma))d\sigma$$

$$+ \int_{\theta}^{\theta+h} R(\sigma)\sin(\theta+h-\sigma)d\sigma + \int_{\theta-h}^{\theta} R(\sigma)\sin(\sigma-(\theta-h))d\sigma$$

$$= 2\int_0^{\theta} R(\sigma)\sin(\theta-\sigma)\cos h\,d\sigma + I_1 + I_2 = 2f(\theta)\cos h + I_1 + I_2.$$

The integrands in I_1 and I_2 are non negative, hence $I_1 + I_2 \geqq 0$ and (1) holds.

(b) *f' absolutely continuous.*

$$f' = \lim_{h\to 0} \frac{1}{h}(f(\theta+h)-f(\theta))$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_0^{\theta+h} R(\sigma)\sin(\theta-\sigma+h)\,d\sigma - \int_0^{\theta} R(\sigma)\sin(\theta-\sigma)d\sigma\right)$$

$$= \lim_{h\to 0} \frac{\sin h}{h}\int_0^{\theta+h} R(\sigma)\cos(\theta-\sigma)d\sigma + \lim_{h\to 0}\frac{\cos h-1}{h}\int_0^{\theta} R(\sigma)\sin(\theta-\sigma)d\sigma$$

$$+ \lim_{h\to 0}\frac{\cos h}{h}\int_{\theta}^{\theta+h} R(\sigma)\sin(\theta-\sigma)d\sigma.$$

But $\lim_{h\to 0}\dfrac{\cos h-1}{h} = 0$ and since $0 \leqq |\sin(\theta-\sigma)| \leqq |h|$ for $|\theta-\sigma| \leqq |h|$ we also have $\lim_{h\to 0}\dfrac{\cos h}{h}\displaystyle\int_{\theta}^{\theta+h} R(\sigma)\sin(\theta-\sigma)d\sigma = 0$. Therefore

$$f' = \int_0^{\theta} R(\sigma)\cos(\theta-\sigma)d\sigma = \cos\theta\int_0^{\theta} R(\sigma)\cos\sigma\,d\sigma + \sin\theta\int_0^{\theta} R(\sigma)\sin\sigma\,d\sigma.$$

The integrals exist, by conditions (iii) and (iv) of Part I, hence f' is absolutely continuous.

(c) $f(\theta)+f(\theta+\pi) = w(\theta)$. Let $v(\theta) = f(\theta)+f(\theta+\pi)$. By (b) we have

$$v'(\theta) = \int_0^{\theta} R(\sigma)\cos(\theta-\sigma)d\sigma + \int_0^{\theta+\pi} R(\sigma)\cos(\theta+\pi-\sigma)d\sigma$$

$$= \cos\theta\int_0^{\theta} R(\sigma)\cos\sigma\,d\sigma + \sin\theta\int_0^{\theta} R(\sigma)\sin\sigma\,d\sigma + \cos(\theta+\pi)\int_0^{\theta+\pi} R(\sigma)\cos\sigma\,d\sigma$$

$$+ \sin(\theta+\pi)\int_0^{\theta+\pi} R(\sigma)\sin\sigma\,d\sigma$$

and $v''(\theta) = R(\theta) + R(\theta + \pi) - v(\theta)$ a.e..

Therefore v and w are two solutions of the differential equation $u''(\theta) + u(\theta) = R(\theta) + R(\theta + \pi)$ a.e.. Both v and w have absolutely continuous derivatives and the same initial conditions since $v(0) = f(0) + f(\pi) = \int_0^\pi R(\sigma)\sin(\pi - \sigma)d\sigma = w(0)$, by condition (iii) of Part I and $v'(0) = f'(0) + f'(\pi) = \int_0^\pi R(\sigma)\cos(\pi - \sigma)d\sigma = w'(0)$, by condition (iv) of Part I. By the uniqueness theorem for differential equations (see [3, Ch. 2]) $v = w$ or $f(\theta) + f(\theta + \pi) = w(\theta)$.

(d) *f is 2π periodic.* This follows immediately from (c).

(e) $R_f(\theta) = R(\theta)$ a.e.. By differentiating

$$f' = \cos\theta \int_0^\theta R(\sigma)\cos\sigma\, d\sigma + \sin\theta \int_0^\theta R(\sigma)\sin\sigma\, d\sigma$$

we obtain

$$f''(\theta) = -\sin\theta \int_0^\theta R(\sigma)\cos\sigma\, d\sigma + R(\theta)\cos^2\theta + \cos\theta \int_0^\theta R(\sigma)\sin\sigma\, d\sigma + R(\theta)\sin^2\theta$$
$$= R(\theta) - f(\theta) \text{ a.e.}.$$

It is clear that equality holds in every θ where $R(\theta)$ is continuous.

By Theorem 1, f is the support function of a convex body K. K is in $\tilde{K}(w)$ by (c) and the radius of curvature of K equals $R(\theta)$ a.e. by (e) and Theorem 3.

The uniqueness (up to translation) of K follows from the fact that every φ with an absolutely continuous derivative which satisfies $\varphi'' + \varphi = 0$ a.e. is of the form $\varphi(\theta) = A\cos\theta + B\sin\theta$ (see [3, Ch. 2]). Hence the general solution of $g''(\theta) + g(\theta) = R(\theta)$ is

$$g(\theta) = \int_0^\theta R(\sigma)\sin(\theta - \sigma)d\sigma + \langle a, u_\theta\rangle = f(\theta) + \langle a, u_\theta\rangle$$

which is the support function of a translate of K.

Now we can prove Part I. By Theorem 2, $f_K \in AC^1$ hence f_K'' exists a.e. and $f_K'(\theta)$ is an integral of f_K''. Therefore f_K'' is measurable and so is $R_{f_K} = f_K'' + f_K$. By Theorem 3, $R_K(\theta) = R_{f_K}(\theta)$ a.e., hence R_K is measurable. Condition (ii) of Part I is obvious by Theorem 3. It is seen by the proof of Part II of this theorem that we may assume that $f_K(\theta) = \int_0^\theta R_K(\sigma)\sin(\theta - \sigma)d\sigma$. Thus we have

$$w(0) = f(0) + f(\pi) = \int_0^\pi R(\sigma)\sin(\pi - \sigma)d\sigma = \int_0^\pi R(\sigma)\sin\sigma\, d\sigma$$

$$w'(0) = f'(0) + f'(\pi) = \int_0^\pi R(\sigma)\cos(\pi - \sigma)d\sigma = -\int_0^\pi R(\sigma)\cos\sigma\, d\sigma$$

which completes the proof.

The width function is not indispensable for Part II (i) (the existence part) of Theorem 4. The following version can be proved in quite a similar way.

THEOREM 4*. *For any measurable nonnegative, 2π periodic function $R(\theta)$ satisfying* $\int_0^{2\pi} R(\theta)\cos\theta\, d\theta = \int_0^{2\pi} R(\theta)\sin\theta\, d\theta = 0$ *there exists a convex body K whose radius of curvature equals $R(\theta)$* a.e. .

However, in general K is not unique. For example if K is any convex polygon, then $R_K(\theta) = 0$ a.e. .

For any function $R(\theta)$ satisfying the conditions of Theorem 4* there is up to translation a unique body K with an AC^1 support function such that $R_K(\theta) = R(\theta)$ a.e. , but having an AC^1 support function has no obvious geometric meaning.

4.

The radius of curvature $R_f = f'' + f$ is additive, and it is natural to examine it in connection with addition of convex bodies.

DEFINITION. A convex body $K \in \tilde{K}(w)$ is said to be extreme in $\tilde{K}(w)$ if $K = \lambda K_1 + (1-\lambda)K_2$, with $K_1, K_2 \in \tilde{K}(w)$ and $0 < \lambda < 1$, implies that K_1 and K_2 are translates of K.

THEOREM 5. *Let $K \in \tilde{K}(w)$. K is extreme in $\tilde{K}(w)$ iff for almost all θ either $R(\theta) = 0$ or $R(\theta + \pi) = 0$.*

PROOF. The radius of curvature is additive, so $K = \lambda K_1 + (1-\lambda)K_2$ implies $R = \lambda R_{K_1} + (1-\lambda)R_{K_2}$. But $0 \leqq R_{K_i}(\theta) \leqq w''(\theta) + w(\theta)$ for all θ and $i = 1,2$ hence if a.e. $R(\theta) = 0$ or $R(\theta) = w''(\theta) + w(\theta)$ then $R_{K_1}(\theta) = R_{K_2}(\theta) = R(\theta)$ a.e.. Part II (ii) of Theorem 4 implies that K_1 and K_2 are translates of K, hence K is extreme in $K(w)$.

Suppose now that $K \in \tilde{K}(w)$, f and R are its support and radius of curvature functions respectively. Let A be a set of positive measure on which $0 < R < w'' + w$. We may assume that $A \subseteq [0,\pi]$ and that there is a positive number ε, such that $\varepsilon < R(\theta) < w''(\theta) + w(\theta) - \varepsilon$ for $\theta \in A$. Consider the linear space $F(A)$ of all real functions on A of the form $a + b\sin\theta + c\cos\theta$, with the inner product

$$\langle f, g\rangle = \int_A f(\theta)g(\theta)d\theta.$$

$F(A)$ is a 3-dimensional inner product space, therefore there exists a function

$g \neq 0$ in $F(A)$ which is orthogonal on A to both $\sin\theta$ and $\cos\theta$. For a sufficiently small positive λ, $|\lambda g(\theta)| < \varepsilon$ for all θ in A.

Define $T(\theta)$, first for $0 \leqq \theta < 2\pi$, by

$$T(\theta) = \begin{cases} \lambda g(\theta) & \text{for} \quad \theta \in A \\ -\lambda g(\theta) & \text{for} \quad \theta \in A + \pi \\ 0 & \text{otherwise} \end{cases}$$

and then extend T to be a 2π periodic function on R.

$|T(\theta)| < \varepsilon$ for all $\theta \in A$ and $T(\theta) = 0$ for $0 \leqq \theta \leqq 2\pi$ outside A and $A + \pi$, hence $0 \leqq R(\theta) \pm T(\theta) \leqq w''(\theta) + w(\theta)$ for all θ. By definition we have $\int_0^\pi T(\theta)\sin\theta\, d\theta = 0$ and $\int_0^\pi T(\theta)\cos\theta\, d\theta = 0$. By Theorem 4, $R + T$ and $R - T$ are a.e. the radius of curvature functions of bodies in $\tilde{K}(w)$. Let K^+ and K^- denote two such bodies. $T \neq 0$ on a set of positive measure hence K^+ and K^- are not translates of K. But K is a translate of $\frac{1}{2}(K^+ + K^-)$, hence K is not extreme in $\tilde{K}(w)$, and the proof is completed.

5.

On first thought it may seem that the extreme bodies in $\tilde{K}(w)$ must have vertices or corner points in almost every direction or its opposite.

A simple example, for the case $w(\theta) = 1$ (where $\tilde{K}(1)$ is the class of sets of constant width 1) shows that this is far from being true. In fact there exists an extreme body in $\tilde{K}(1)$ whose radius of curvature takes both values 0 and 1 on sets of positive measure in every interval. Such a body is smooth, that is, it has no vertex (interval of support directions with one fixed point of support).

In order to construct the support function of K we first divide the interval $[0, \frac{1}{2})$ into two measurable sets A_1, B_1 in such a way that every interval $[a, b]$ with $0 \leqq a < b \leqq \frac{1}{2}$ intersects both A_1 and B_1 in sets of positive measure (see [5, p. 99]). Define $A_2 = B_1 + \frac{1}{2}$, $B_2 = A_1 + \frac{1}{2}$, and $A = A_1 \cup A_2$, $B = B_1 \cup B_2 \cup \{1\}$. Thus $\mu(A) = \mu(B) = \frac{1}{2}$.

Now define a homeomorphism φ: $[0,1] \to [0, \frac{1}{2}\pi]$ by $\varphi(\xi) = \operatorname{arc}\cos(1 - \xi)$. We have $\varphi^{-1}(\theta) = 1 - \cos\theta$.

Define a function $R(\theta)$, first in the interval $[0, \frac{1}{2}\pi]$, by

$$R(\theta) = \begin{cases} 1 & \text{if} \quad \theta \in \varphi(A) \\ 0 & \text{if} \quad \theta \in \varphi(B). \end{cases}$$

Extend the definition to $[0, \pi)$ by $R(\frac{1}{2}\pi + \theta) = R(\frac{1}{2}\pi - \theta)$ for $0 \leqq \theta < \frac{1}{2}\pi$, and then to the whole real line by: $R(\theta + \pi) = 1 - R(\theta)$. R is an even function with respect to $\frac{1}{2}\pi$ on $(0,\pi)$, hence $\int_0^\pi R(\theta)\cos\theta\, d\theta = 0$ and

$$\int_0^\pi R(\theta)\sin\theta\, d\theta = 2\int_0^{\frac{1}{2}\pi} R(\theta)\sin\theta\, d\theta = 2\int_{\varphi(A)} d(1-\cos\theta)$$
$$= 2\int_{\varphi(A)} d(\varphi^{-1}(\theta)) = 2\int_A d\xi = 1.$$

By Theorem 4, $R(\theta)$ is a.e. the radius of curvature of a convex body K in $K(1)$. K is smooth because if K had a vertex then $R_K(\theta)$ would vanish on a whole interval.

6.

Our last application of Theorem 4 is in the theory of bodies of constant width. A Reuleaux polygon is a body in $K(1)$ whose boundary consists of a finite number of circular arcs of radius 1. Its radius of curvature function takes only the values 1 and 0, each on a finite number of intervals between 0 and 2π. Reuleaux polygons of width 1 are known to be dense in $\tilde{K}(1)$ with respect to the Hausdorff metric (see [1]). Here we prove a somewhat stronger version of this density theorem.

THEOREM 6. *Let K be in $\tilde{K}(1)$. For each $\varepsilon > 0$ there is a Reuleaux polygon K_ε with support function f_ε satisfying the following conditions.*

(i) $\max_{0 \leqq \theta \leqq 2\pi} |f_K(\theta) - f_\varepsilon(\theta)| \leqq \varepsilon.$

(ii) $\max_{0 \leqq \theta \leqq 2\pi} |f'_k(\theta) - f'_\varepsilon(\theta)| \leqq \varepsilon.$

(iii) *K_ε has no more than $2[\pi/\varepsilon(2)^{\frac{1}{2}}] + 3$ sides, and all its vertices with support directions between 0 and π lie on the boundary of K.*

PROOF. Let $n = [\pi/\varepsilon(2)^{\frac{1}{2}}] + 1$ and let $a_i = i\pi/n$ $(0 \leqq i \leqq n)$. In the interval $[a_{i-1}, a_i]$ there exists a one-parameter family of subintervals $[b_i(\lambda), c_i(\lambda)]$ for $0 \leqq \lambda \leqq 1$ with $b_i(0) = a_{i-1}$, $c_i(1) = a_i$, $b_i(\lambda)$, and $c_i(\lambda)$ continuous, nondecreasing functions of λ, such that

$$\int_{a_{i-1}}^{a_i} R(\theta)\sin\theta\, d\theta = \int_{b_i(\lambda)}^{c_i(\lambda)} \sin\theta\, d\theta \tag{11}$$

for all $0 \leqq \lambda \leqq 1$.

Define a new measure v on $[0,\pi]$ by $v(E) = \int_E \sin\theta \, d\theta$. For any integrable function R we have

$$\int_E R(\theta)\cos\theta \, d\theta = \int_E R(\theta)\operatorname{cotg}\theta \, dv(\theta) \tag{12}$$

and

$$v[b_i(\lambda), c_i(\lambda)] = \int_{a_{i-1}}^{a_i} R(\theta) dv \text{ for } 0 \leqq \lambda \leqq 1. \tag{13}$$ Now $\operatorname{cotg}\theta$ is decreasing in $[0, \pi]$, hence

$$\int_{b_i(0)}^{c_i(0)} \operatorname{cotg}\theta \, dv(\theta) \geqq \int_{a_{i-1}}^{a_i} R(\theta)\operatorname{cotg}\theta \, dv(\theta) \geqq \int_{b_i(1)}^{c_i(1)} \operatorname{cotg}\theta \, dv(\theta).$$

Thus, for a suitable choice of λ we have

$$\int_{a_{i-1}}^{a_i} R(\theta) \operatorname{cotg}\theta \, dv(\theta) = \int_{b_i(\lambda)}^{c_i(\lambda)} \operatorname{cotg}\theta \, dv(\theta). \tag{14}$$

Let $b_i = b_i(\lambda)$ and $c_i = c_i(\lambda)$ for that choice of λ. Then for each m ($0 \leqq m \leqq n$) we have, by (12) and (14)

$$\int_0^{a_m} R(\theta) \cos\theta \, d\theta = \sum_{i=1}^{m} \int_{b_i}^{c_i} \cos\theta \, d\theta \tag{15}$$

and by (11)

$$\int_0^{a_m} R(\theta)\sin\theta \, d\theta = \sum_{i=1}^{m} \int_{b_i}^{c_i} \sin\theta \, d\theta. \tag{16}$$

Define a function $R_\varepsilon(\theta)$, first in the interval $[0,\pi)$ by

$$R_\varepsilon(\theta) = \begin{cases} 1 & \text{if } b_i \leqq \theta \leqq c_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

and extend the definition to all real θ by $R_\varepsilon(\theta + \pi) = 1 - R_\varepsilon(\theta)$. By (15) and (16) for $m = n$ it is clear that $R_\varepsilon(\theta)$ satisfies the conditions of Theorem 4 with $w = 1$. Therefore $R_\varepsilon(\theta)$ is a.e. the radius of curvature function of a convex body K_ε with support function $f_\varepsilon(\theta) = \int_0^\theta R_\varepsilon(\sigma) \sin(\theta - \sigma) d\sigma$. But $R_\varepsilon(\theta)$ is piecewise continuous, hence $R_\varepsilon(\theta)$ is the radius of curvature of K_ε except for a finite number of directions (see the proof of Part II (v) of Theorem 4). It is readily seen that the points 0, b_i, c_i, π, $b_i + \pi$, $c_i + \pi$, 2π divide the interval $[0, 2\pi]$ into an even number, at most $4n + 2$, of subintervals and that $R_\varepsilon(\theta)$ assumes on these intervals the constant values 0 and 1 alternately. Since the solution of the differential equation $f'' + f = R$

on any interval is unique up to translation, it follows that the intervals with $R = 1$ correspond to circular arcs of radius 1 on the boundary of K_ε and the intervals with $R = 0$ correspond to vertices of K. Thus K_ε is clearly a Reuleaux polygon with at most $2n + 1 = 2[\pi/\varepsilon(2)^{\frac{1}{2}}] + 3$ sides.

For each $0 \leqq \theta \leqq \pi$ there is some i for which

$$|\theta - a_i| \leqq \tfrac{1}{2}|a_i - a_{i-1}| = \pi/2n \leqq \varepsilon(2)^{-\frac{1}{2}}.$$

By (16) we have

$$I(\theta) = \left|\int_0^\theta (R_K(\sigma) - R_\varepsilon(\sigma))\sin\sigma\, d\sigma\right| = \left|\int_\theta^{a_i} (R_K(\sigma) - R_\varepsilon(\sigma))\sin\sigma\, d\sigma\right| \leqq \varepsilon(2)^{-\frac{1}{2}},$$

and similarly by (15)

$$J(\theta) = \left|\int_0^\theta (R_K(\sigma) - R_\varepsilon(\sigma))\cos\sigma\, d\sigma\right| \leqq \varepsilon(2)^{-\frac{1}{2}}.$$

Since K can be replaced by any translate of K, we may assume that $f_K(\theta) = \int_0^\theta R_K(\sigma)\sin(\theta - \sigma)d\sigma$, and for $0 \leqq \theta \leqq \pi$

$$|f_K(\theta) - f_\varepsilon(\theta)| = \left|\int_0^\theta (R_K(\sigma) - R_\varepsilon(\sigma))\sin(\theta - \sigma)d\sigma\right| \leqq |\sin\theta|J(\theta) + |\cos\theta|\, I(\theta)$$

$$\leqq \varepsilon(2)^{-\frac{1}{2}}(|\sin\theta| + |\cos\theta|) \leqq \varepsilon$$

and similarly $|f'_K(\theta) - f'_\varepsilon(\theta)| \leqq |\sin\theta|\, I(\theta) + |\cos\theta|J(\theta) \leqq \varepsilon$.

These inequalities hold also for $\pi \leqq \theta \leqq 2\pi$ since

$$f_K(\theta + \pi) = 1 - f_K(\theta);\ f_\varepsilon(\theta + \pi) = 1 - f_\varepsilon(\theta).$$

The point of contact of a convex body $K \subset E^2$ with a support line which has an outer normal u_θ is completely determined by $f'_K(\theta)$ and $f_K(\theta)$. (In [4, pp. 56–7] it is shown that the point of contact is determined by the support function $H(K, u)$, and its partial derivatives. The connection to $f'_K(\theta)$ and $f_K(\theta)$ is obvious.) By the construction, $f_K(a_i) = f_\varepsilon(a_i)$ and $f'_K(a_i) = f'_\varepsilon(a_i)$. It follows that all the vertices of K with support directions between 0 and π lie on the boundary of K.

Remark. This paper was originally written for the special case $w = 1$ (constant width). The generalization was motivated by Ruth Silverman's characterization of the indecomposable bodies in $\tilde{K}(w)$ [9]. Unfortunately, the characterization given in [9] is incorrect, and there is a mistake in the proof of one of the lemmas ([9, Lem. 6]).

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